Propagation Mode and Scattering Loss of a Two-Dimensional Dielectric Waveguide with Gradual Distribution of Refractive Index

YASUHARU SUEMATSU, MEMBER, IEEE, AND KAZUHITO FURUYA

Abstract—An analytical discussion of the mode property and the scattering loss of a two-dimensional dielectric waveguide with gradual refractive-index distribution in the transverse direction is presented. To describe scattering loss, a transverse correlation as well as an axial correlation of the irregular variation of the refractive index have been used.

The field distribution, the group delay, and the maximum film thickness of a single-mode waveguide scarcely depends on the shape of the distribution. The maximum value of the film thickness in the single-mode transmission region optimizes the scattering loss and the energy confinement. The scattering loss of a waveguide with a gradual index distribution is smaller than that of a three-layer waveguide when the transverse correlation is small, but it is not much altered when the transverse correlation is large.

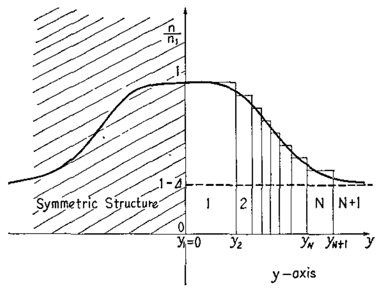

Fig. 1. Approximation with a staircase function. The waveguide whose refractive index changes gradually is approximated by N-step multilayer structure.

I. Introduction

DIELECTRIC WAVEGUIDES are considered to be very promising at optical frequencies [1]. Optical fibers [2], integrated optics [3], [4], and other possibilities [5] are making use of this type of guide. The properties of the dielectric waveguides have been discussed mainly under the condition that the dielectric constant or the refractive index changes abruptly at the core boundary, except for the case of lens-like media [6].

In actual cases, the distribution of refractive index near the boundary of the guide is sometimes gradual because of the diffusion mechanism of the constructing materials. But the distribution of refractive index in this case is sharp compared with that of the lens-like medium.

In a dielectric waveguide, besides the propagation constant, scattering loss that comes from the irregularities of the boundary is an important factor to characterize the guided properties. The field distributions and the propagation constants in a cylindrical guide of a gradual distribution of refractive index were treated in [7]. The scattering loss from the one abrupt irregular boundary was given in [8]–[10].

In this paper we present an analytical discussion of the mode property and the scattering loss of a two-dimensional dielectric waveguide with a gradual distribution of refractive index. The distribution of refractive index has been approximated by a staircase function. A transverse matrix representation has been introduced to treat it. To describe scattering loss, a transverse correlation as well as an axial correlation of the irregular variation of the refractive index have been used.

II. Characteristic-Mode Analysis by Means of F-Matrix

It is assumed that the waveguide is a two-dimensional structure, the refractive index is constant along the z axis, and is distributed symmetrically along the y axis with respect to $y=0$. The waveguide whose refractive index changes gradually along the y axis is approximated by an N-step multilayer structure, as shown in Fig. 1.

If the refractive index changes spatially, the gradients of the refractive index enter into the wave equations [11]. Instead of taking these gradients into account, we divide the space into $(N+2)$ layers, and make use of the boundary conditions at the $N+1$ boundaries to satisfy Maxwell's equations. In order to treat the TE and TM modes in a similar form, the following notations are used.

$$\Phi = \begin{Bmatrix} E_x \\ \eta H_x \end{Bmatrix}$$

$$\Psi = \begin{Bmatrix} -j\eta H_z \\ jE_z \end{Bmatrix}, \qquad \begin{Bmatrix} \text{for TE wave} \\ \text{for TM wave} \end{Bmatrix} \tag{1}$$

where $\eta = (\sqrt{\mu/\epsilon_0})/n_1$ and n_1 is the refractive index at the center of the guide. The upper line in the parentheses applies if TE wave is to be analyzed, while the

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The authors are with the Department of Electronics, Tokyo Institute of Technology, O-okayama, Meguro-Ku, Tokyo, Japan 152.

lower line belongs to TM wave. In the ith layer the solution of Maxwell's equations is represented by means of the transverse F-matrix as follows.

$$\begin{bmatrix}\Phi(y)\\ \Psi(y)\end{bmatrix} = [F_i(y-y_0)]\begin{bmatrix}\Phi(y_0)\\ \Psi(y_0)\end{bmatrix}, \quad \begin{matrix} y_i \leqq y_0 \leqq y_{i+1} \\ y_i \leqq y \leqq y_{i+1}\end{matrix} \tag{2}$$

where

$$[F_i(y)] \equiv \begin{bmatrix} \cos \gamma_i y & (-k_1\zeta_i/\gamma_i)\sin \gamma_i y \\ (\gamma_i/k_1\zeta_i)\sin\gamma_i y & \cos\gamma_i y \end{bmatrix}$$

$$\gamma_i^2 = k_1^2(n_i/n_1)^2 - \beta^2$$

$$k_1^2 = \omega^2 n_1^2 \epsilon_0 \mu$$

$$\zeta_i = \begin{Bmatrix} 1 \\ (n_i/n_1)^2 \end{Bmatrix}, \quad \begin{Bmatrix} \text{TE wave} \\ \text{TM wave}\end{Bmatrix}$$

and β is the propagation constant along the z axis. The modes of a symmetrical dielectric waveguide have definite parity, which means they are either even or odd functions and therefore satisfy the following relations at the center of the guide.

$$\Psi(0) = 0, \quad \text{for even modes}$$

$$\Phi(0) = 0, \quad \text{for odd modes.} \tag{3}$$

At boundaries between two adjoining layers, field components parallel to the boundary surface, or perpendicular to the y axis, must be continuous because of the boundary condition. Therefore, the following equation represented by the product of matrices $\hat{\prod}$ must be satisfied.

$$\begin{bmatrix}\Phi(0)\\ \Psi(0)\end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_t & D_t\end{bmatrix}\begin{bmatrix}\Phi(y_{N+1})\\ \Psi(y_{N+1})\end{bmatrix} \tag{4}$$

where

$$\begin{bmatrix} A_t & B_t \\ C_t & D_t\end{bmatrix} \equiv \hat{\prod_{i=1}^{N}} [F_i(-d_i)]$$

and d_i is the thickness of the ith layer. The condition for a guided mode is that the field decays exponentially in the outermost layer, or the $(N+1)$th layer. Therefore, γ_{N+1} is imaginary, and so the range of the propagation constant β of the guided mode is

$$k_1^2(1-2\Delta) < \beta^2 < k_1^2 \tag{5}$$

where

$$\Delta = (n_1^2 - n_{N+1}^2)/2n_1^2 \simeq (n_1 - n_{N+1})/n_1, \quad \text{for } (n_1 - n_{N+1}) \ll n_1.$$

And at $y = y_{N+1}$ the field components must satisfy the following equation

$$\frac{\Phi(y_{N+1})}{\Psi(y_{N+1})} = \frac{k_1\zeta_{N+1}}{(\gamma_{N+1}/j)}. \tag{6}$$

From (3), (4), and (6) the determinative equation for the guided mode is

$$\zeta_{N+1}C_t k_1 + D_t(\gamma_{N+1}/j) = 0, \quad \text{for even modes}$$

$$\zeta_{N+1}A_t k_1 + B_t(\gamma_{N+1}/j) = 0, \quad \text{for odd modes.} \tag{7}$$

For the radiation modes the eigenvalue or the propagation constant β is continuous, and its range is given by

$$0 < \beta^2 < k_1^2(1-2\Delta). \tag{8}$$

The set of radiation modes contains degenerated pairs; that is, two modes belong to the same propagation constant and are orthogonal to each other. The orthogonality is held between the transverse electric field and the magnetic field, and the normalization constant is selected as $P = 1/(\omega\mu)$ to simplify the analysis (Appendix II).

III. Analysis of the Scattering Loss

The scattering loss caused by the deviated distribution of the refractive index or the imperfection of the boundary is determined as the mode-conversion loss in a manner similar to [8], using the characteristic modes determined in Section II.

A. Mode Conversion Caused by a Deviated-Index Distribution

The waveguide with a deviation in refractive index is described by a refractive-index distribution

$$n^2(y,z) = n_0^2(y) + \delta n^2(y,z) \tag{9}$$

where $n_0(y)$ describes the ideal dielectric waveguide whose modes were given in Section II, and the additional term δn^2 describes how the index deviates from the perfect distribution. It is possible to express any field distribution on the waveguide with the deviated index distribution by the expansion

$$\Phi(y,z) = \sum_{n=0}^{\nu} c_n(z)\,|\Phi_n\rangle + \sum \int g(z;\beta')\,|\Phi(\beta')\rangle\, d\gamma_{N+1}' \tag{10}$$

where "kets" $|\,\rangle$ represent characteristic modes of the waveguide (Appendix I), and $(\nu+1)$ is the number of guided modes. The summation sign in front of the integral indicates summation over degenerate modes, even, and odd modes.

To obtain differential equations for the expansion coefficients we substitute (10) into the perturbed wave equation that contains the term δn^2. Multiplying the resulting equation by $(\beta_m/\eta)\langle\Phi_m|(1/\zeta)$ or $(\beta'/\eta)\langle\Phi(\beta')|(1/\zeta)$ from the left, and using the orthonormality relation (32) and the fact that $|\Phi_n\rangle$ and $|\Phi(\beta)\rangle$ are the characteristic modes of the perfect guide, leads to

$$\left[\frac{\partial^2}{\partial z^2} - 2j\begin{Bmatrix}\beta_m\\ \beta'\end{Bmatrix}\frac{\partial}{\partial z}\right]\begin{Bmatrix} c_m \\ g(\beta')\end{Bmatrix} = \begin{Bmatrix} F_m(z) \\ G(z;\beta')\end{Bmatrix} \tag{11}$$

with

$$\begin{Bmatrix} F_m(z) \\ G(z;\beta) \end{Bmatrix} = -\frac{(k_1/n_1)^2}{2\omega\mu P}\begin{Bmatrix} \beta_m \\ \beta \end{Bmatrix}\sum_n c_n \begin{Bmatrix} \left\langle \Phi_m \middle| \frac{\delta n^2}{\zeta} \middle| \Phi_n \right\rangle \\ \left\langle \Phi(\beta) \middle| \frac{\delta n^2}{\zeta} \middle| \Phi_n \right\rangle \end{Bmatrix} + \sum \int g(\beta') \begin{Bmatrix} \left\langle \Phi_m \middle| \frac{\delta n^2}{\zeta} \middle| \Phi(\beta') \right\rangle \\ \left\langle \Phi(\beta) \middle| \frac{\delta n^2}{\zeta} \middle| \Phi(\beta') \right\rangle \end{Bmatrix} d\gamma_{N+1}'. \quad (12)$$

For the purpose of obtaining perturbation solutions of (11), an integral form of this equation is useful. In this form we can separate it into two parts, where one is associated with the wave traveling in the positive z direction and the other in the negative. Therefore, we introduce the notation

$$c_m(z) = c_m^{(+)}(z) + c_m^{(-)}(z)$$
$$g(z;\beta) = g^{(+)}(z;\beta) + g^{(-)}(z;\beta). \quad (13)$$

The constant occurring in the integral form is determined from initial conditions (Appendix III).

In order to solve the integral equation we employ a first-order perturbation method by using $c_m(0)$ instead of $c_m(z)$, and $g(0;\beta)$ instead of $g(z;\beta)$ in the integrand of the integral equation. And we use

$$c_m(0) = \delta_{0m}$$
$$g(0;\beta) = 0. \quad (14)$$

Now solutions for $c_m^{(+)}(z)$, $c_m^{(-)}(z)$, $g^{(+)}(z;\beta)$, and $g^{(-)}(z;\beta)$ are obtained in first-order approximation (Appendix III). Then the power loss ΔP of the incident mode due to the mode conversion caused by the imperfect section, $0 \leqq z \leqq L$, is given by

$$\Delta P/P = \sum_{n=1}^{\nu} |c_n^{(+)}(L)|^2 + \sum_{n=1}^{\nu} |c_n^{(-)}(0)|^2 + \sum \int (|g^{(+)}(L;\beta')|^2 + |g^{(-)}(0;\beta')|^2) d\gamma_{N+1}'. \quad (15)$$

B. Statistical Treatment of Index Deviations

If a definite (deterministic) deviation of the index distribution were given, the relative loss of a guide could be calculated from (15). If certain statistical properties of the deviation, such as the correlation lengths of the fluctuation, are known, we can determine the average of the relative loss taken over an ensemble of statistically identical systems.

Since the refractive index changes gradually, two-dimensional correlations between the deviation of the index at two points are taken into account.

The integration over y from $-\infty$ to ∞ defined by the bracket $\langle\Phi|\delta n^2/\zeta|\Phi_0\rangle$ in (12) and (35) is replaced by the summation over the multilayers, which is used in the characteristic-mode analysis of the waveguide. For example, from (12) or (35), $F_m(z)$ is

$$F_m(z) \simeq -\frac{\beta_m(k_1/n_1)^2}{2\omega\mu P}\sum_{i=-(N+1)}^{N+1} \Phi_m(y_i)\frac{\delta n_i^2}{\zeta_i}\Phi_0(y_i) \cdot d_i \exp\{j(\beta_m - \beta_0)z\}. \quad (16)$$

Then from (34) (using (16), for example) the ensemble average of the square magnitudes of the mth forward-traveling guided mode is

$$\overline{\langle|c_m^{(+)}(L)|^2\rangle} = -\frac{(k_1/n_1)^4}{16(\omega\mu P)^2}\int_0^L dz\int_0^L dz' \exp\{j(\beta_m-\beta_0)(z-z')\} \cdot \sum_{i=-(N+1)}^{N+1}\sum_{j=-(N+1)}^{N+1} \overline{\langle\delta n_i^2(z)\delta n_j^2(z')\rangle}\Phi_m(y_i)\Phi_m(y_j) \cdot \Phi_0(y_i)\Phi_0(y_j)d_id_j/\zeta_i\zeta_j, \qquad (y_{-i} = -y_i). \quad (17)$$

The factor $\overline{\langle\delta n_i^2(z)\delta n_j^2(z')\rangle}$ represents the ensemble average of the product of the dielectric-constant deviation δn^2 in the ith layer at $z=z$ by that in the jth layer at $z=z'$. We may assume that the index deviation δn^2 is caused by a local shift δy of the index distribution along the y axis. Then the index deviation δn^2 and the local shift δy are related by

$$\delta n_i^2 = -\frac{d}{dy}(n^2)\bigg|_{y=y_i}\delta y_i. \quad (18)$$

Using N layers, we divide the difference between the dielectric constant of the center layer and that of the outermost layer into N equal steps. Therefore,

$$-\frac{d}{dy}(n^2)\bigg|_{y=y_i} = \frac{(2\Delta)n_1^2}{d_iN}, \qquad (i = 2, 3, \cdots, N+1)$$

and

$$-\frac{d}{dy}(n^2)\bigg|_{y=y_1} = 0. \quad (19)$$

From (18) and (19),

$$\overline{\langle\delta n_i^2(z)\delta n_j^2(z')\rangle}\, d_id_j = \frac{(4\Delta^2)n_1^4}{N^2}\overline{\langle\delta y_i(z)\delta y_j(z')\rangle}. \quad (20)$$

We assume the following form for the correlation function

$$\overline{\langle\delta y_i(z)\delta y_j(z')\rangle} = A^2 \exp(-|y_i - y_j|/B_y) \cdot \exp(-|z-z'|/B_z). \quad (21)$$

A, the rms deviation of the index distribution, is assumed to be independent of y. And B_y, B_z are the corre-

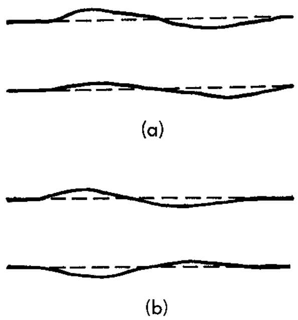

Fig. 2. (a) Plus-type correlation. (b) Minus-type correlation. Under the plus- (minus-) type correlation, the direction of the local shift of the index distribution on one side is the same as (opposite to) that on the other side.

lation length in the transverse direction and longitudinal direction, respectively. The last exponential term in (21) was adopted by Marcuse [8] for his analysis, which contains no transverse correlation.

Substituting (20) and (21) into (17), integrating over z and z', and assuming that $L \gg B_z$, we obtain

$$\overline{\left\langle \left| c_m^{(\pm)} \begin{Bmatrix} L \\ 0 \end{Bmatrix} \right|^2 \right\rangle} = A^2 L k_1^4 (2\Delta/N)^2 \frac{B_z}{\{B_z(\beta_0 \mp \beta_m)\}^2 + 1} S_m, \quad (m \neq 0)$$

with

$$S_m = \frac{1}{4(\omega\mu P)^2} \sum_{i=2}^{N+1} \sum_{j=2}^{N+1} [\Phi_m(y_i)\Phi_m(y_j)\Phi_0(y_i)\Phi_0(y_j) \cdot \exp(-|y_i - y_j|/B_y)/(\zeta_i\zeta_j) \pm \{-\Phi_m(y_i)\Phi_m(-y_j)\Phi_0(y_i)\Phi_0(-y_j) \cdot \exp(-(y_i + y_j)/B_y)/(\zeta_i\zeta_j)\}]. \quad (22)$$

The first term in the bracket represents the contribution to the mode coupling of the correlation between the index deviations at two points of the dielectric waveguide, which are on the same side with respect to $y=0$. The second term is the contribution of the correlation between the index deviations at two points, which are located on opposite sides with respect to $y=0$. For the sign of the second term in (22), we consider two cases, the plus-type and the minus-type correlation. The plus (minus) sign corresponds to the plus- (minus-) type correlation. For the plus- (minus-) type correlation, the direction of the local shift of the index distribution on one side of the dielectric waveguide is the same as (opposite to) that on the other side (Fig. 2).

The corresponding expressions for the other modes are derived similarly.

$$\overline{\left\langle \left| g^{(\pm)} \begin{Bmatrix} L \\ 0 \end{Bmatrix} ; \beta) \right|^2 \right\rangle} = A^2 L k_1^4 (2\Delta/N)^2 \frac{B_z}{\{B_z(\beta_0 \mp \beta)\}^2 + 1} S(\beta). \quad (23)$$

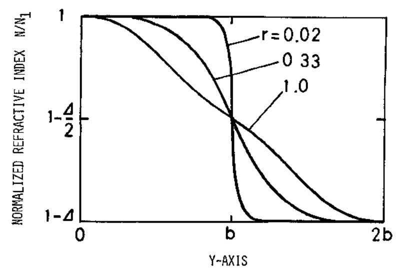

Fig. 3. Gradual refractive-index distribution. The value of r is smaller, the slope of the curve at $y=b$ is steeper, but the area under the curve is independent of r.

$S(\beta)$ is obtained by replacing Φ_m with $\Phi(\beta)$ in S_m, see (22).

If we consider the index deviation δn^2 to be caused by wall deviations, and assume deviations at two points to be related by (21), we obtain the same results. However, there is the difficulty that the wall deviation δy_i or A cannot exceed the thickness d_i.

The ensemble average of (15) is calculated from the following equation, using (22) and (23).

$$\langle \Delta P/P \rangle = \sum_{n=1}^{\nu} \overline{\langle | c_n^{(+)}(L) |^2 \rangle} + \sum_{n=0}^{\nu} \overline{\langle | c_n^{(-)}(0) |^2 \rangle} + \sum \int \left(\overline{\langle | g^{(+)}(L;\beta') |^2 \rangle} + \overline{\langle | g^{(-)}(0;\beta') |^2 \rangle} \right) d\gamma_{N+1}'. \quad (24)$$

IV. Numerical Results

A. Gradual Refractive-Index Distribution

In order to investigate how the shape of refractive-index distribution affects characteristic modes, propagation constants, group velocities, the maximum film thickness that provides single-mode transmission, and the scattering loss caused by the deviation of the refractive-index distribution, we assume the refractive-index distribution to be

$$n(y)/n_1 = 1 - \frac{\Delta}{2} \exp\left(\frac{1}{r}\right) \exp\left(-\frac{b}{ry}\right), \quad 0 < |y| \le b$$

$$n(y)/n_1 = 1 - \Delta + \frac{\Delta}{2} \exp\left(\frac{1}{r}\right) \cdot \exp\left(-\frac{b}{r(2b-y)}\right), \quad b < |y| < 2b$$

$$n(y)/n_1 = 1 - \Delta, \quad |y| > 2b \quad (25)$$

for small Δ. The parameter r determines the shape of the distribution (Fig. 3). For a small value of r, the slope of the n–y curve at $y=b$ is steep, and in the limit $r \rightarrow 0$ the multilayer dielectric waveguide becomes a symmetric three-layer waveguide.

For a constant value of $2b$, the area between the n–y curve and the line $n=1-\Delta$ is independent of the param-

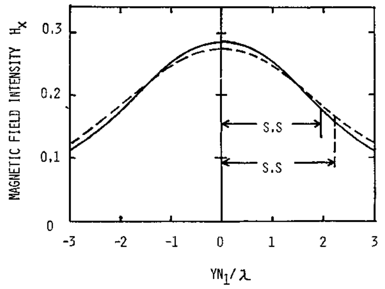

Fig. 4. Field distributions ($\Delta=0.01$, $2b=3.02\lambda/n_1$, $n=8$). Solid curve corresponds to a steep distribution ($r=0.02$) and dashed curve to a gradual distribution ($r=1.0$).

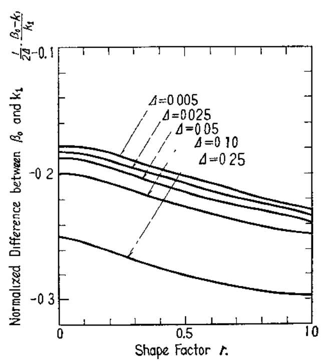

Fig. 5. Propagation constants of fundamental modes become small as distributions become gradual ($N=8$).

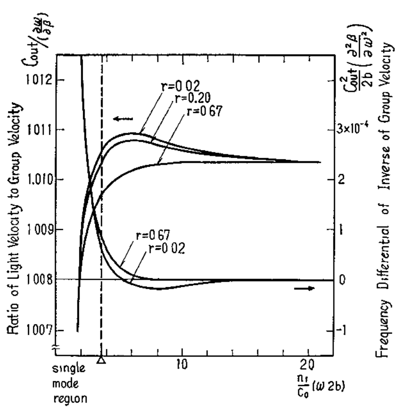

Fig. 6. Frequency characteristics of group velocities of fundamental modes and their first derivatives ($N=8$, $\Delta=0.01$).

eter r. Therefore, $2b$ is the equivalent film thickness of the waveguide with gradual index distribution.

B. Characteristic Modes, Propagation Constants, and Group Velocities

Figs. 4–6 show numerical evaluations for field distributions, propagation constants, and group velocities of fundamental TM modes.

Fig. 4 applies to waveguides with $\Delta=0.01$, $2b=3.02 \cdot (\lambda/n_1)$, and $N=8$. It is shown that the more gradual the index distribution becomes, the wider the mode becomes. But the change is not so remarkable. The spot size, i.e., the width of the field distribution where the intensity is smaller than that at the center by exp $(-1/2)$, corresponding to $r=0.02$, is 2.03 (λ/n_1), and the spot size corresponding to $r=1.0$ is 2.24 (λ/n_1). So the change in the spot size due to the change in r is as small as 10 percent.

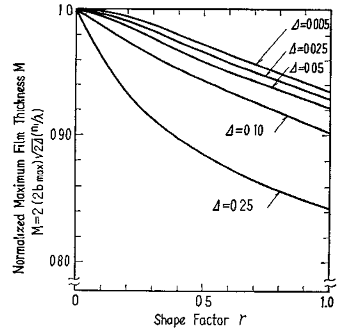

Fig. 7. The maximum film thickness of a single-mode waveguide becomes small as r becomes large ($N=8$).

As Fig. 5 shows, the propagation constants of the fundamental modes become small as the distributions become gradual.

Fig. 6 shows frequency characteristics of group velocities of fundamental modes and their first derivatives. But for single-mode transmission, there is little change due to the change of r.

C. The Maximum Film Thickness of Single-Mode Waveguides

For many applications, single-mode dielectric waveguides may be of great interest, and, furthermore, for the reason mentioned in part D, the maximum value of the film thickness that provides single-mode transmission is important.

In the case of the dielectric waveguide with abrupt change in the refractive-index distribution at $y=|b|$, the maximum film thickness is

$$2b = 1/(2\sqrt{2\Delta})(\lambda/n_1).$$

And,

$$2b_{\max} = M(\lambda/n_1)/(2\sqrt{2\Delta}) \tag{26}$$

where $2b_{\max}$ is the maximum equivalent film thickness of single-mode waveguide for the multilayer waveguide. The value of M depends on the value of r, as shown in Fig. 7. M decreases with increasing r.

D. Field Distributions of Waveguides with Function of Film Thickness

As the film thickness or the width of the core of the symmetric three-layer waveguide changes, the field distribution of the fundamental mode changes, as shown in Fig. 8.

In the case of $2b=0$, the wave is not confined at all. The spot size of the fundamental mode decreases with increasing $2b$. It becomes minimum at some value of $2b$; for example, at $2b=2(\lambda/n_1)$ for $\Delta=0.01$.

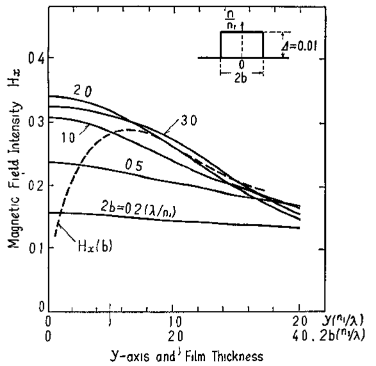

Fig. 8. Field distributions corresponding to various film thickness and the field intensity at the boundary versus film thickness $2b$ ($N=1$, $\Delta=0.01$).

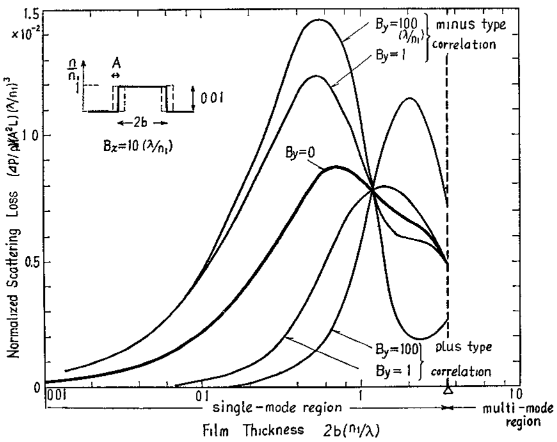

Fig. 9. Scattering loss versus film thickness. The peak shifts to the right for the plus-type correlation and to the left for the minus-type correlation ($N=1$, $\Delta=0.01$, $B_z=10\lambda/n_1$).

The value of the field intensity at the boundary, which has an intimate relation to the scattering loss caused by wall imperfections, depends on $2b$, as shown in Fig. 8.

E. Scattering Loss versus Film Thickness

Fig. 9 applies to a waveguide with $B_z=10(\lambda/n_1)$. This value provides the maximum scattering loss for $\Delta=0.01$, as will be shown in part F.

In the case where there is no correlation between the deviations on opposite sides, the scattering loss peaks at $2b=2b_p$; for example, $2b_p=0.7$ (λ/n_1) for $\Delta=0.01$. If $2b<2b_p$, the smaller $2b$ becomes the smaller the loss becomes (Fig. 9), but the confining character of the guide also becomes weaker (Fig. 8). However, if $2b>2b_p$, the loss decreases monotonically as $2b$ increases; but at $2b=2b_{\max}$, the mode coupling between the fundamental mode and the second guided mode starts.

Taking the transverse correlation into account, the peak $2b_p$ shifts to the right for the plus-type correlation and to the left for the minus-type correlation.

These phenomena are interpreted with the help of (22) and (24). In (24), the third term contributes the most. Therefore,

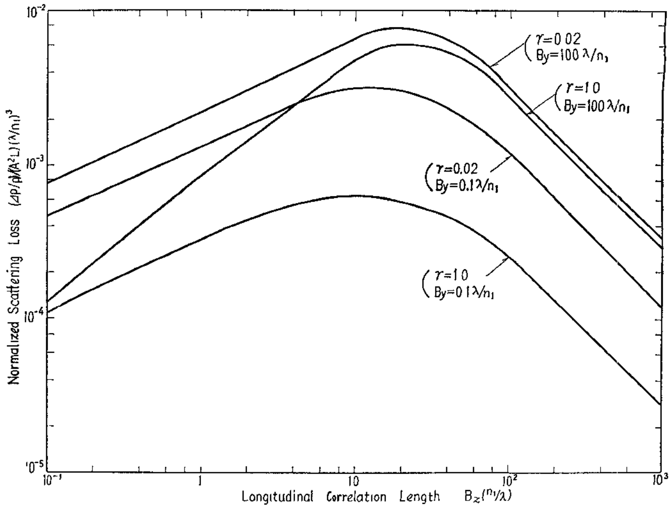

Fig. 10. Scattering loss versus correlation length B_z. The loss peaks at around $B_z=10\lambda/n_1$ ($N=8$, $\Delta=0.01$, $2b=2b_{\max}\simeq3.5\lambda/n_1$).

$$\overline{\left\langle\frac{\Delta P}{P}\right\rangle} \propto \int d\gamma_{N+1}' \frac{B_z}{\{B_z(\beta_0-\beta')\}^2+1} \cdot \left[2\Phi_0^2(b)\left\{(\Phi_e^2(b)+\Phi_d^2(b)) \pm(-\Phi_e^2(b)+\Phi_d^2(b))\exp(-2b/B_y)\right\}\right] \quad (27)$$

where Φ_e and Φ_d indicate the even mode and the odd mode, respectively, and the plus sign and minus sign in front of the second term corresponds to the plus correlation and minus correlation, respectively. As long as B_z is not so small, the integral in (27) contributes to the loss only at $\beta'\simeq1-\Delta$ due to the first factor. For $2b$ smaller than a certain value, $\Phi_e^2(b)>\Phi_d^2(b)$, while for $2b$ larger than a certain value, $\Phi_e^2(d)<\Phi_d^2(b)$. Thus the sign of the correlation term in (27) changes at a certain value of $2b$ and the peak of the curve shifts.

It is also clear from (27) that if the transverse correlation length B_y is long enough that $\exp(-2b/B_y)\simeq1$, and in the case of the plus-type correlation $\Phi_e^2(b)$ is cancelled out, the fundamental mode does not couple with even radiation modes but couples only with odd radiation modes. The opposite is true for minus-type correlation.

In order to realize dielectric waveguides with low scattering loss caused by wall imperfections, the value of $2b$ should be optimized. The optimum value of $2b$ is a little smaller than $2b_{\max}$. For small values of $2b$, the spot size is large so that the loss caused by bends of the guide is also large.

F. Scattering Loss versus Correlation Length B_z, B_y

Fig. 10 applies to a waveguide with $\Delta=0.01$, $2b=2b_{\max}$, $N=8$ as an example. According to this figure, the loss peaks at around $B_z=10(\lambda/n_1)$.

Fig. 11 applies to a waveguide with $\Delta=0.01$, $2b=2b_{\max}$, $N=8$, $B_z=10(\lambda/n_1)$ as an example. The loss increases monotonically with the transverse correlation length B_y.

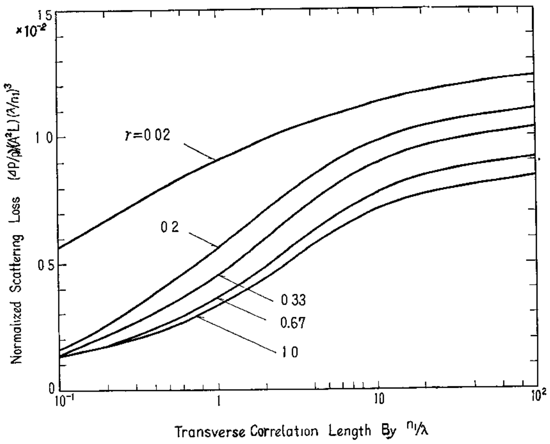

Fig. 11. Scattering loss versus correlation length B_y. The loss increases monotonously with the transverse correlation length $B_y (N=8, \Delta=0.01, 2b=2b_{max}, B_z=10\lambda/n_1)$.

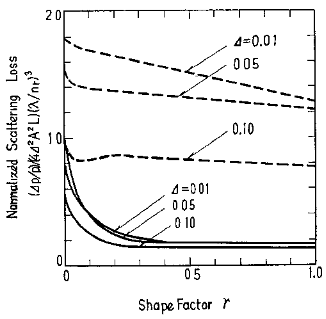

Fig. 12. Scattering loss versus shape factor r. The loss reduction due to the gradual index distribution is appreciable when B_y is small ($B_y=0.1\lambda/n_1$, solid line), but is not so appreciable when B_y is large ($B_y=100\lambda/n_1$, dashed line) ($N=8$, $\Delta=0.01$, $2b=2b_{max}$).

G. Scattering Loss versus Shape Factor r

The scattering loss caused by the deviation from the perfect index distribution decreases as the value of r increases, as shown in Fig. 12. The loss reduction due to the gradual index distribution is appreciable only when B_y is small. This effect is interpreted qualitatively with the help of (22) and (24). If B_z is not too small, the term

$$\frac{B_z}{\{B_z(\beta_0 - \beta')\}^2 + 1}$$

occurring in (22) gives contributions to the integral of (24) only in the neighborhood of $\beta' = 1 - \Delta$. For radiation modes with propagation constants $\beta' \simeq 1 - \Delta$, the field intensity varies slowly along the y axis, which is also true for the fundamental mode. We assume that the value of $2b$ is small so that

$$\Phi_0(\pm y_2) \simeq \Phi_0(\pm y_3) \simeq \cdots \simeq \Phi_0(\pm y_{N+1}) \simeq \Phi_0(b)$$
$$\Phi(\pm y_2; \beta') \simeq \Phi(\pm y_3; \beta') \simeq \cdots \simeq \Phi(\pm y_{N+1}; \beta') \simeq \Phi(\pm b; \beta'). \quad (28)$$

From (23), using (28) and $\zeta_i \simeq 1$, and omitting the term of the correlation between the deviations on opposite sides

$$\overline{\langle |g^{(+)}(L)|^2 \rangle} \propto \left(\frac{1}{N}\right)^2 \sum_{i=2}^{N+1} \sum_{j=2}^{N+1} \exp\left(-|y_i - y_j|/B_y\right) = F(N; B_y). \quad (29)$$

If N is so small that

$$\min\left(|y_i - y_{i+1}|\right) \gg B_y, \qquad (i = 2, 3, \cdots, N) \quad (30)$$

then

$$F(N; B_y) = \left(\frac{1}{N}\right)^2 \sum_{i=2}^{N+1} \sum_{j=2}^{N+1} \delta_{ij} = \frac{1}{N}.$$

As far as (30) holds, increasing N, the loss decreases in inverse proportion to N. If N becomes so large that

$$\sum_{j=2}^{N+1} \exp\left(-|y_i - y_j|/B_y\right) = N f_i(B_y, y_{N+1} - y_2) \gg 1$$

then $f_i(B_y, y_{N+1} - y_2)$ increases monotonically with B_y, and decreases with $(y_{N+1} - y_2)$. With further approximation that $f_i(B_y, y_{N+1} - y_2)$ is independent of i, we obtain the following relation

$$F(N; B_y) = f(B_y, y_{N+1} - y_2). \quad (31)$$

Equation (31) shows that the scattering loss of an N-layer dielectric waveguide converges with larger N to a value that depends on the transverse correlation length B_y and $(y_{N+1} - y_2)$, and therefore on the shape parameter r.

We evaluated the loss for waveguides with $N=8$, $N=20$, and $N=60$, but there is not an appreciable difference among them.

In the case of $B_y = 0.1(\lambda/n_1)$, the loss corresponding to $r=1$ is smaller than that corresponding to $r=0$ by the factor 1/8.

V. Conclusion

We have analyzed the properties of a dielectric waveguide with gradual refractive-index distribution in transverse direction, using the transverse F-matrix.

The field distribution of a fundamental mode, group delay, and the maximum film thickness of a single-mode waveguide scarcely depend on the shape of the index distribution.

Taking the transverse, as well as the longitudinal, correlation length into account, we have calculated the scattering loss caused by deviations of the index distribution.

The maximum value of the film thickness in the single-mode transmission region optimizes the scattering loss and the energy confinement. The loss of a waveguide with a gradual index distribution and transverse correlation length $B_y = 0.1(\lambda/n_1)$ is only 1/8 of the loss of a three-layer waveguide. However, in the case of $B_y = 100(\lambda/n_1)$, there is no significant difference between the former and the latter.

Appendix I
"Bracket" Notation

For simplicity of calculation, "bra" and "ket" notations are used for the characteristic modes of dielectric waveguide. Definitions are as follows.

$$|\Phi_n\rangle \equiv \Phi_n(y)\exp(-j\beta_n z)$$

$$|\Phi(\beta)\rangle \equiv \Phi(y;\beta)\exp(-j\beta z)$$

$$\langle\Phi_n| \equiv (\Phi_n(y)\exp(-j\beta_n z))^*$$

$$\langle\Phi(\beta)| \equiv (\Phi(y;\beta)\exp(-j\beta z))^*$$

$$\langle\Phi_n|A|\Phi(\beta)\rangle \equiv \int_{-\infty}^{\infty}\Phi_n^*(y)A\Phi(y;\beta)\,dy\exp(j(\beta_n-\beta)z)$$

where A is an operater or a function.

Appendix II
Normalization

The normalization is defined using "bracket" notation (Appendix I).

$$P\delta_{mn} = \frac{\beta_n}{2\omega\mu}\left\langle\Phi_m\left|\frac{1}{\zeta(y)}\right|\Phi_n\right\rangle, \quad \text{for guided modes}$$

$$P\delta(\gamma_{N+1}-\gamma_{N+1}') = \frac{\beta}{2\omega\mu}\left\langle\Phi(\beta)\left|\frac{1}{\zeta(y)}\right|\Phi(\beta')\right\rangle, \quad \text{for radiation modes.} \tag{32}$$

Modes are normalized by following normalization constants

$$C_n = \left[\frac{\beta_n}{2\omega\mu P}\left\{\sum_{i=1}^{N}\left\{\frac{1}{\zeta_i}\Phi^2(y_i)\left(d_i+\frac{1}{2\gamma_i}\sin 2\gamma_i d_i\right) + \zeta_i\frac{k_1^2}{\gamma_i^2}\Psi^2(y_i)\left(d_i-\frac{1}{2\gamma_i}\sin 2\gamma_i d_i\right) + \frac{k_1}{\gamma_i^2}\Phi(y_i)\Psi(y_i)(\cos 2\gamma_i d_i - 1)\right\} + \frac{j}{\gamma_{N+1}\zeta_{N+1}}\Phi^2(y_{N+1})\right\}\right]^{-1/2}, \quad \text{for guided modes}$$

$$C(\beta) = \left[\frac{\pi\beta}{2\omega\mu P}\left\{\frac{1}{\zeta_{N+1}}\Phi^2(y_{N+1}) + \frac{k_1^2\zeta_{N+1}}{\gamma_{N+1}^2}\Psi^2(y_{N+1})\right\}\right]^{-1/2}, \quad \text{for radiation modes.} \tag{33}$$

Appendix III
Expansion Coefficients

The integration constant is determined from the initial conditions that only the lowest order guided mode is incident on imperfect waveguide at $z=0$, and that at $z=L$ the waveguide is connected to a perfect guide so that at $z=L$ there are no waves traveling in negative z direction. Then integral equations are

$$\begin{Bmatrix} c_m^{(+)}(z) \\ g^{(+)}(z;\beta)\end{Bmatrix} = -\frac{1}{2j\beta}\int_0^z \begin{Bmatrix} F_m(u) \\ G(u;\beta)\end{Bmatrix} du, \qquad (m\neq 0)$$

$$\begin{Bmatrix} c_m^{(-)}(z) \\ g^{(-)}(z;\beta)\end{Bmatrix} = -\frac{1}{2j\beta}\int_z^L \exp(-2j\beta u)\begin{Bmatrix} F_m \\ G(\beta)\end{Bmatrix} du \cdot \exp(2j\beta z). \tag{34}$$

In first-order perturbation the integrand in (34) is approximated by

$$\begin{Bmatrix} F_m \\ G(\beta)\end{Bmatrix} \simeq -\frac{\beta(k_1/n_1)^2}{2\omega\mu P}\begin{Bmatrix} \left\langle\Phi_m\left|\frac{\delta n^2}{\zeta}\right|\Phi_0\right\rangle \\ \left\langle\Phi(\beta)\left|\frac{\delta n^2}{\zeta}\right|\Phi_0\right\rangle\end{Bmatrix}. \tag{35}$$

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